A remark on differentiability of Cauchy horizons

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Abstract

In a recent paper Królak and Beem [1] have shown differentiability of Cauchy horizons at all points of multiplicity one. In this note we give a simpler proof of this result.

1 Introduction

A question of current interest is that of differentiability of various horizons that occur in general relativity. Recall that in [2] it was shown that there exist Cauchy horizons, as well as black hole event horizons, which are non differentiable on a dense set. In that reference it was also shown that

1. Cauchy horizons are differentiable at all interior points of their generators;
2. Cauchy horizons are not differentiable at all end points of generators of multiplicity larger than one.

(Recall that the multiplicity of an end point of a generator is defined as the number (perhaps infinite) of generators which end at this point.) These results leave open the question of differentiability of a Cauchy horizon at end points of multiplicity one. In a recent paper Królak and Beem [1] have settled this issue, showing differentiability of Cauchy horizons at those points. In this note we give a simpler proof of this result. Actually, motivated by the question of differentiability of black hole horizons, we will prove differentiability of a somewhat larger class of hypersurfaces, *cf.* Theorem 2.3 below.

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2 Statements and proofs

Before proving our main result, Theorem 2.3, we need the following preliminary result:

Lemma 2.1 *Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set, suppose that $f \in C^{0,1}(\mathcal{U})$ and consider*

$$\mathcal{H} = \{t = f(\vec{x}), \vec{x} \in \mathcal{U}\}\,.$$

Then $\mathcal{H}$ is differentiable at $\vec{x}_0 \in \mathcal{U}$ if and only if there exists a hypersurface $T \subset \mathbb{R} \times \mathbb{R}^n$ such that for every sequence $(f(\vec{x}_0) + \epsilon_i w_i, x_0 + \epsilon_i \vec{v}_i) \in \mathcal{H}$ for which $\epsilon_i \to 0, w_i \to w$ and $\vec{v}_i \to \vec{v}$ we have $(w, \vec{v}) \in T$.

Proof: $\Rightarrow$ By a slight abuse of notation consider f to be a function on $\mathbb{R} \times \mathcal{U}$ satisfying $\partial f/\partial t = 0$, where t is the variable running along the $\mathbb{R}$ factor. Let dt be the derivative of t at $(f(\vec{x}_0), \vec{x}_0)$, and let df be the derivative of f at $(f(\vec{x}_0), \vec{x}_0)$, then $T = \ker(dt - df)$.

$\Leftarrow$ Let (e_0, e_i) denote the standard basis of $\mathbb{R} \times \mathbb{R}^n$ and let (f^0, f^i) be the corresponding dual basis. Consider any $\alpha \in (\mathbb{R} \times \mathbb{R}^n)^*$ such that $T = \ker \alpha$, thus α can be written in the form $\alpha_r f^r$ (summation convention); note that $\alpha \neq 0$ since codim $T = 1$. Let $\vec{v} \in \mathbb{R}^n$ such that $\sum (v^i)^2 = 1$ and let ϵ_i be any sequence converging to zero; consider the sequence $(f(\vec{x}_0 + \epsilon_i \vec{v}), \vec{x}_0 + \epsilon_i \vec{v}) \to (f(\vec{x}_0), \vec{x}_0)$. Since f is Lipschitz continuous we have $|f(\vec{x}_0 + \epsilon_i \vec{v}) - f(\vec{x}_0)| \leq L\epsilon_i$ and compactness of $[-L, L]$ implies that there exists a subsequence ϵ_{i_j} such that

$$\left(\frac{f(\vec{x}_0 + \epsilon_{i_j} \vec{v}) - f(\vec{x}_0)}{\epsilon_{i_j}}, \vec{v} \right)$$

converges to $(v^0, \vec{v})$. By hypothesis $(v^0, \vec{v}) \in T$, thus $\alpha_0 v^0 + \alpha_i v^i = 0$. Note that $\alpha_0 v^0 = 0$ implies $\alpha_i v^i = 0$ and, hence, $\alpha_i = 0$ by arbitrariness of v^i. It follows that we can always normalize α so that $\alpha_0 = 1$ and we get $v^0 = -\alpha_i v^i$. We thus have

$$\lim_{j \to \infty} \frac{f(\vec{x}_0 + \epsilon_{i_j} \vec{v}) - f(\vec{x}_0)}{\epsilon_{i_j}} = -\alpha_i v^i. \tag{2.1}$$

As the right–hand–side of (2.1) does not depend upon the sequence ϵ_i, we must actually have

$$\lim_{\epsilon \to 0} \frac{f(\vec{x}_0 + \epsilon \vec{v}) - f(\vec{x}_0)}{\epsilon} = -\alpha_i v^i.$$

This can be rewritten as

$$f(\vec{x}_0 + \epsilon\vec{v}) = f(\vec{x}_0) - \epsilon\alpha_i v^i + o(\epsilon),$$

which is what had to be established. $\square$

Lemma 2.1 allows us to give a simple proof of the main result of Beem and Królak [1]; recall that $\mathcal{N}_p(\mathcal{H})$ denotes [2] the set of null semi-tangents at p to a Cauchy horizon $\mathcal{H}$, *i.e.*, the set of vectors tangent to some generator of $\mathcal{H}$ through p, p being possibly (but not necessarily) an end point of such a generator, oriented to the past for future Cauchy horizons, and to the future for past Cauchy horizons. We also normalize those generators to length one with respect to some fixed auxiliary Riemannian metric.

Theorem 2.2 (Beem and Królak [1]) *Let $p \in \mathcal{H}$ be such that $\#\mathcal{N}_p(\mathcal{H}) = 1$. Then $\mathcal{H}$ is differentiable at p.*

Theorem 2.2 follows immediately from the following, somewhat more general statement:

Theorem 2.3 *Let $\mathcal{H}$ be a topological hypersurface satisfying the following:*

1. *$\mathcal{H}$ is* locally achronal, i.e., *for any $p \in \mathcal{H}$ there exists a neighborhood $\mathcal{O}$ of p such that $\mathcal{H} \cap \mathcal{O}$ is achronal in the space–time $(\mathcal{O}, g|_{\mathcal{O}})$.*
2. *Every point p of $\mathcal{H}$ is either an interior point of a null geodesic $\Gamma \subset \mathcal{H}$, or a future endpoint thereof. Such Γ's will be called generators of $\mathcal{H}$.*

Then $\mathcal{H}$ is differentiable at every point p which belongs to only one generator of $\mathcal{H}$.

Proof: Let $p_i \in \mathcal{H}$ be any sequence such that $p_i \to p$, and let $\gamma_i \in \mathcal{N}_{p_i}(\mathcal{H})$; we have $\gamma_i \to \gamma \in \mathcal{N}_p(\mathcal{H})$ (*cf., e.g.*, [2, Lemma 3.1], together with the argument of the proof of Proposition 3.3 there). In normal coordinates centered at p we can write

$$p_i = p + d_i v_i, \qquad 0 \le d_i \to 0,$$

where the length of the v_i's has been normalized to 1 using some auxiliary Riemannian metric M. For $\epsilon, \epsilon_i \ge 0$ let $p_i(\epsilon_i) \in \mathcal{H}$, respectively $p(\epsilon) \in \mathcal{H}$, denote the point lying an affine distance ϵ_i, respectively ϵ, on the null generator of $\mathcal{H}$ with semi-tangent γ_i, respectively γ. From $\gamma_i \to \gamma$ we have $\gamma_i - \gamma = o(1)$, and from the fact that in normal coordinates null geodesics through $p + d_i v_i$ at affine distance ϵ_i differ from straight lines by terms which are $o(d_i + \epsilon_i)$ we obtain

$$\begin{aligned} p_i(\epsilon_i) &= p + d_i v_i + \epsilon_i \gamma_i + o(d_i + \epsilon_i), \\ p(\epsilon) &= p + \epsilon\gamma. \end{aligned}$$

Let η be the Minkowski metric, consider the quantity

$$\begin{aligned} A &= \eta(p_i(\epsilon_i) - p, p_i(\epsilon_i) - p) \\ &= d_i^2 \eta(v_i, v_i) + 2 d_i \epsilon_i \eta(v_i, \gamma) + o((d_i + \epsilon_i)^2). \end{aligned} \tag{2.2}$$

Suppose that $v_i \to v$, and suppose, first, that $\eta(v, v) < 0$. Equation (2.2) with $\epsilon_i = 0$ gives $A < 0$ for i large enough. It follows that the coordinate line through p and $p_i(0) = p_i$ is timelike which contradicts achronality of $\mathcal{H}$, hence

$$\eta(v, v) \geq 0. \tag{2.3}$$

Suppose, next, that $\eta(v, v) > 0$ and $\eta(v, \gamma) < 0$. In that case Equation (2.2) with $\epsilon_i = \dfrac{\eta(v, v)}{\mid \eta(v, \gamma) \mid} d_i$ gives $A < 0$. It follows that the coordinate line through p and $p_i(\epsilon_i)$ is timelike which is again impossible, so that

$$\eta(v, \gamma) \geq 0. \tag{2.4}$$

If $\eta(v, v) = 0$, Equation (2.2) with $\epsilon_i = d_i$ leads similarly to (2.4).

To show that the inequality (2.4) has to be an equality, consider the coordinate lines starting at $p + v_i$ and ending at $p + \epsilon_i \gamma$:

$$[0, 1] \ni s \to \Gamma_i(s) = p + (1 - s) d_i v_i + s \epsilon_i \gamma \ .$$

On $\Gamma_i(s)$ we have

$$\begin{aligned} g(v_i, v_i) &= g(v, v) + o(1) = \eta(v, v) + o(1) \ , \\ g(\gamma, v_i) &= g(\gamma, v) + o(1) = \eta(\gamma, v) + o(1) \ , \\ g(\gamma, \gamma) &= g(\gamma, \gamma) + o(1) = o(1) \ , \end{aligned}$$

which implies

$$g(\frac{d\Gamma_i}{ds}, \frac{d\Gamma_i}{ds}) = d_i^2 \eta(v, v) - 2 \epsilon_i d_i \eta(v, \gamma) + o((d_i + \epsilon_i)^2) \ . \tag{2.5}$$

Note that Equation (2.5) differs from Equation (2.2) only by the sign of the $\eta(v_i, \gamma)$ terms, so that a similar analysis shows that Γ_i will be timelike for i large enough unless

$$\eta(v, \gamma) = 0.$$

It follows that $v \in T \equiv \gamma^\perp$. The local achronality of $\mathcal{H}$ implies that $\mathcal{H}$ is Lipschitz, and differentiability of $\mathcal{H}$ at p follows now from Lemma 2.1. □

References

[1] J. Beem and A. Królak, *Cauchy horizon endpoints and differentiability*, Jour. Math. Phys. (1998), in press, gr-qc/9709046.

[2] P.T. Chruściel and G.J. Galloway, *Horizons non–differentiable on dense sets*, Commun. Math. Phys. (1998), 449–470, gr-qc/9611032.